AB Calculus Review Sheet #4

Review the formulas that may be used to calculate area between curves, volumes by cross section, and volumes of revolution.

Review numerical methods such as trapezoidal method and problems involving data.

Review accumulating rate problems.

Multiple Choice Questions:

1. The area in square units bounded by $x^2 = y^2(4 - y^2)$ is

(A) $\frac{16}{3}$ (B) $\frac{32}{3}$ (C) $\frac{8}{3}$ (D) $\frac{64}{15}$ (E) 4

2. Suppose the following is a table of values for $y = f(x)$, given that f is continuous on $[1,5]$:

x	1	2	3	4	5
y	1.62	4.15	7.50	9.00	12.13

If a trapezoid sum is used, with $n = 4$, then the area bounded by the curve and the x-axis, from $x = 1$ to $x = 5$ is equal to

(A) 6.88 (B) 13.76 (C) 20.30 (D) 25.73 (E) 27.53

3. Find the volume of the solid generated when the region contained in the quadrilateral with vertices at (2,0), (2,2), (4,0), and (4,4) is revolved about the x-axis.

(A) $\frac{56\pi}{3}$ (B) $\frac{28\pi}{3}$ (C) $\frac{92\pi}{3}$ (D) $\frac{112\pi}{3}$ (E) none of these

4. The volume of the solid generated when the region bounded by $y = 3x - x^2$ and $y = x$ is revolved about the x-axis is

(A) $\pi\int_0^{3/2}\left[(3x-x^2)^2 - x^2\right]dx$ (B) $\pi\int_0^2(9x^2 - 6x^3)dx$ (C) $\pi\int_0^2\left[(3x-x^2)^2 - x^2\right]dx$

(D) $\pi\int_0^3\left[(3x-x^2)^2 - x^4\right]dx$ (E) $\pi\int_0^3(2x - x^2)^2 dx$

5. The base of a solid is the region bounded by the parabola $x^2 = 8y$ and the line $y = 4$, and each plane section perpendicular to the y-axis is an equilateral triangle. The volume of the solid is

(A) $\frac{64\sqrt{3}}{3}$ (B) $64\sqrt{3}$ (C) $32\sqrt{3}$ (D) 32 (E) none of these

6. The base of the solid is a circle of radius a, and every plane section perpendicular to a diameter is a square. The solid has volume

(A) $\frac{8a^3}{3}$ (B) $2\pi a^3$ (C) $4\pi a^3$ (D) $\frac{16a^3}{3}$ (E) $\frac{8\pi a^3}{3}$

7. Oil is leaking from a tanker at the rate of $1000e^{-0.3t}$ gallons per hour, where t is given in hours. The total number of gallons of oil that will leak out during the next 8 hours is approximately

(A) 1271 (B) 3031 (C) 3161 (D) 4323 (E) 11,023

8. If a factory continuously dumps pollutants into a river at the rate of $\frac{\sqrt{t}}{180}$ tons per day (t is measured in days, then the amount dumped after 7 weeks is approximately

(A) 0.07 tons (B) 0.90 tons (C) 1.55 tons

(D) 1.90 tons (E) 1.27 tons

9. How long will it take to release 9 tons of pollutant if the rate at which pollutant is being released is $te^{-0.3t}$ tons per week?

(A) 10.2 weeks (B) 11.0 weeks (C) 12.1 weeks

(D) 12.9 weeks (E) none of these

10. A rumor spreads through a town at the rate of $(t^2 + 10t)$ new people per day. Approximately how many people hear the rumor during the second week after it was first heard?

(A) 1535 (B) 1894 (C) 2000 (D) 2219 (E) none of these

Free Response Questions

1. The rate of consumption of cola in the United States is given by $S(t) = Ce^{kt}$, where S is measured in billions of gallons per year and t is measured in years from the beginning of 1980.

(a) The consumption rate doubles every 5 years and the consumption rate at the beginning of 1980 was 6 billion gallons per year. Find C and k.

(b) Find the average rate of consumption of cola over the 10-year time period beginning January 1, 1983. Indicate units of measure.

(c) Use the trapezoidal rule with four equal subdivisions to estimate $\int_5^7 S(t)\,dt$.

(d) Using correct units, explain the meaning of $\int_5^7 S(t)\,dt$ in terms of cola consumption.

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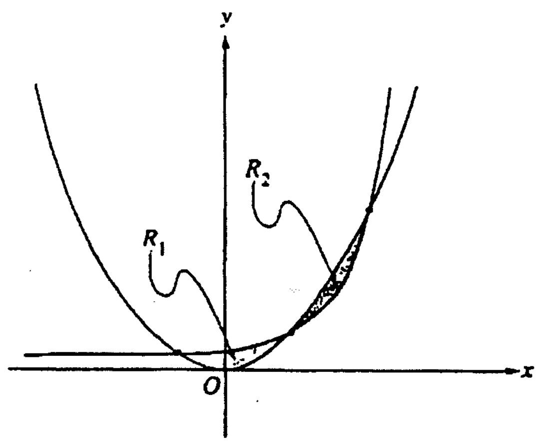
2. Let R be the region in the first quadrant under the graph of $y = \frac{1}{\sqrt{x}}$ for $4 \le x \le 9$.

(a) Find the area of R.

(b) If the line $x = k$ divides the region R into two regions of equal area, what is the value of k?

(c) Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis are squares.



Note: Figure not drawn to scale.

3. The shaded regions R_1 and R_2 shown above are enclosed by the graphs of $f(x) = x^2$ and $g(x) = 2^x$.

(a) Find the x- and y-coordinates of the three points of intersection of the graphs of f and g.

(b) Without using absolute value, set u an expression involving one or more integrals that gives the total area enclosed by the graphs of f and g. Do not evaluate.

(c) Without using absolute value, set up an expression involving one or more integrals that gives the volume of the solid generated by revolving the region R_1 about the line $y = 5$. Do not evaluate.

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4. (a) Determine and evaluate a definite integral for which

$$\frac{1}{40}\left[0^3 + 2(0.05)^3 + 2(0.1)^3 + \cdots + 2(1.95)^3 + (2)^3\right] \text{ is a trapezoidal approximation.}$$

(b) Which is greater, the integral or the trapezoidal approximation? Why? (Answer the question without having to calculate the answers for either part.)

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5. The rate of consumption of oil in the United States during the 1980s (in billions of barrels per year) modeled by the function $C = 27.08e^{t/25}$, where t is the number of years after January 1, 1980. Find the total consumption of oil in the United States from January 1, 1980 to January 1, 1990.

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6. The rate at which your home consumes electricity is measured in kilowatts. If your home consumes electricity at the rate of 1 kilowatts per hour, you will be charged for 1 "kilowatt-hour" of electricity. Suppose that the average consumption rate for a certain home is modeled by the function $C(t) = 3.9 - 2.4\sin\left(\frac{\pi t}{12}\right)$ where $C(t)$ is measured in kilowatts and t is the number of hours past midnight. Find the amount of electricity in kilowatt-hours consumed in your home on an average day.

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7. Population density measures the number of people per square mile inhabiting a given living area. Washerton's population density, which decreases as you move away from the city's center, can be approximated by the function $D(r) = 10{,}000(2 - r)$ at a distance r miles from the city's center.

   a. If the population density approaches zero at the edge of the city, what is the city's radius?

   b. A thin ring around the center of the city has thickness Δr and radius r. If you straighten the ring out, it suggests a rectangular strip. Approximately what is the area of the ring?

   c. Explain why the population of the ring in part (b) is approximately $P(r) = 10{,}000(2 - r)2\pi r\Delta r$.

   d. Estimate the total population of Washerton by setting up and evaluating a definite integral. Note: Population $= \sum \text{Density} \times \text{Change in Area}$.

Review Sheet #4

1. B	6. D
2. E	7. B
3. A	8. E
4. C	9. A
5. B	10. A

1. 1996 AB3
2. 1996 AB2
3. 1995 AB4
4. See attached
5. See attached
6. See attached
7. See attached

1996 AB

QUESTION #3 / STANDARDS

AB-3, BC-3

1. The rate of consumption of cola in the United States is given by $S(t) = Ce^{kt}$, where S is measured in billions of gallons per year and t is measured in years from the beginning of 1980.

(a) The consumption rate doubles every 5 years and the consumption rate at the beginning of 1980 was 6 billion gallons per year. Find C and k.

(b) Find the average rate of consumption of cola over the 10-year time period beginning January 1, 1983. Indicate units of measure.

(c) Use the trapezoidal rule with four equal subdivisions to estimate $\int_5^7 S(t)\,dt$.

(d) Using correct units, explain the meaning of $\int_5^7 S(t)\,dt$ in terms of cola consumption.

(a) $S(t) = Ce^{kt}$

$S(0) = 6 \implies C = 6$

$S(5) = 12 \implies 12 = 6e^{5k}$

$2 = e^{5k}$

$k = \dfrac{\ln 2}{5}$ (0.138 or 0.139)

3 $\begin{cases} 1: C = 6 \\ 1: 12 = 6e^{5k} \\ 1: k = \dfrac{\ln 2}{5} \end{cases}$

(b) average rate $= \dfrac{1}{13-3}\displaystyle\int_3^{13} 6e^{\frac{\ln 2}{5}t}\,dt$

$= \dfrac{3}{\ln 2}\left[e^{2.6\ln 2} - e^{0.6\ln 2}\right]$ billion gal/yr

(19.680 billion gal/yr)

3 $\begin{cases} 1: \text{uses } [3,13] \text{ and divides by } 13-3 \\ 1: \text{integrand} \\ 1: \text{answer with units} \\ \quad 0/1 \text{ if not } \dfrac{1}{b-a}\displaystyle\int_a^b S(t)\,dt \end{cases}$

(c) $\int_5^7 S(t)\,dt$

$\doteq \dfrac{1}{4}[S(5) + 2S(5.5) + 2S(6) + 2S(6.5) + S(7)]$

1 $\begin{cases} \text{Trapezoidal rule with } S, \\ n = 4, \text{ interval } [5,7] \end{cases}$

(d) This gives the total consumption, in billions of gallons, during the years 1985 and 1986.

2 $\begin{cases} 1: \text{total consumption in a time period} \\ 1: \begin{cases} \text{correct time period} \\ \text{liquid measure} \end{cases} \end{cases}$

0/2 for "rate of consumption"

Question 2

In part (a), the student is asked to find the area of a region. In part (b), the student is asked to find the equation of the vertical line that divides the region in half. Part (c) asks for the volume of the solid whose base is the given region and whose cross sections are squares. It is expected that the student will use a calculator to graph the region to help visualize the problem. The student is permitted, but not required, to use a calculator to compute the required definite integrals.

2. Let R be the region in the first quadrant under the graph of $y = \dfrac{1}{\sqrt{x}}$ for $4 \le x \le 9$.

(a) Find the area of R.

(b) If the line $x = k$ divides the region R into two regions of equal area, what is the value of k?

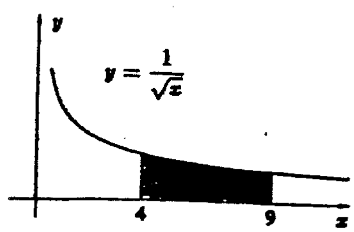
(c) Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis are squares.

Solution

Scoring Scale

Points

(a) $\displaystyle\int_4^9 \frac{dx}{\sqrt{x}} = 2$



$$3 \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \\ \quad 0/1 \text{ if integrand is incorrect} \end{cases}$$

(b) $\displaystyle\int_4^k \frac{dx}{\sqrt{x}} = 1$

$$2\sqrt{x}\Big|_4^k = 1$$

$$2\sqrt{k} - 2\sqrt{4} = 1$$

$$k = \frac{25}{4}$$

$$\left(\text{or } \int_k^9 \frac{dx}{\sqrt{x}} = 1 \text{ or } \int_4^k \frac{dx}{\sqrt{x}} = \int_k^9 \frac{dx}{\sqrt{x}}\right)$$

$$3 \begin{cases} 1: \displaystyle\int_4^k \frac{dx}{\sqrt{x}} \text{ or } \int_k^9 \frac{dx}{\sqrt{x}} \\ 1: \text{equation involving the two halves of } R \\ 1: \text{answer} \\ \quad 0/1 \text{ if answer from equation not involving relevant areas} \end{cases}$$

(c) $\displaystyle \text{volume} = \int_4^9 \left(\frac{1}{\sqrt{x}}\right)^2 dx$

$$= \int_4^9 \frac{dx}{x} = \ln x\Big|_4^9 = \ln\frac{9}{4}$$

(or 0.811)

$$3 \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \\ \quad 0/1 \text{ if integrand is incorrect} \end{cases}$$

3

1995 AB4

Solution	Points	*Scoring Scale*
(a) (2, 4) (4, 16) (–0.767, 0.588) or (–0.766, 0.588)	2	$\langle -1 \rangle$ each incorrect or missing pair Note: max 1/2 if x coordinates only
(b) $\int_{-0.767}^{2} (2^x - x^2)\,dx + \int_{2}^{4} (x^2 - 2^x)\,dx$	4	2: each definite integral 1: limits 1: Integrand, w/o absolute values $\langle -1 \rangle$ Integrals considered but not added Note: 1/4 for $\int_{-0.767}^{4} \lvert x^2 - 2^x \rvert\,dx$ Note: max 1/4 dealing only with one of R_1 or R_2
(c) $\pi \int_{-0.767}^{2} ((5 - x^2)^2 - (5 - 2^x)^2)\,dx$	3	1: limits and π 2: Integrand $\langle -1 \rangle$ reversal $\langle -1 \rangle$ uses absolute value $\langle -1 \rangle$ with additional integral

Alternates

(b) $\int_{0}^{0.588} 2\sqrt{y}\,dy + \int_{0.588}^{4} \left(\sqrt{y} - \frac{\ln y}{\ln 2}\right) dy + \int_{4}^{16} \left(\frac{\ln y}{\ln 2} - \sqrt{y}\right) dy$	4	2: area of R_1 1: limits 1: Integrand 2: area of R_2 1: limits 1: Integrand
(c) $2\pi \int_{0}^{0.588} (5 - y)\, 2\sqrt{y}\,dy + 2\pi \int_{0.588}^{4} (5 - y)\left(\sqrt{y} - \frac{\ln y}{\ln 2}\right) dy$	3	1: limits and 2π 2: 1 for each integrand

Review Sheet #4 Answers

→ (4) $\frac{1}{40}\left[0^3+2(0.05)^3+2(.1)^3+\dots+2(1.95)^3+2^3\right]$

(a) $\Delta x = .05$ $\qquad \frac{1}{40} = \frac{1}{2}\Delta x$

$f(x) = x^3$

$$\int_0^2 x^3\,dx = \left.\frac{x^4}{4}\right|_0^2 = \frac{16}{4} = 4$$

(b) Since $f(x)=x^3$ is concave up, the trapezoidal "lines" will fall above the curve; therefore, the trapezoidal approximation will be larger,

→ (5) $$\int_0^{10} 27.08e^{\frac{1}{25}t} = 27.08(25)e^{\frac{1}{25}t}\Big|_0^{11} = 332.965 \text{ billion barrels}$$

The US will use 332.965 billion barrels over the 10 yr. period

→ (6) Rate of Consumption $= C(t) = 3.9 - 2.4\sin\left(\frac{\pi t}{12}\right)$

Note $C(t) > 0$ for all t.

$$\int_0^{24} 3.9 - 2.4\sin\left(\frac{\pi t}{12}\right) dt =$$

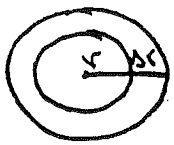
(or) $$3.9t + \frac{2.4\cdot 12}{\pi}\cos\left(\frac{\pi t}{12}\right)\Big|_0^{24} = 93.6 \text{ kilowatt-hours.}$$

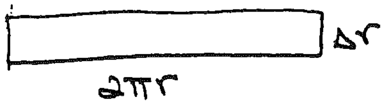
My home uses an average of 93.6 kilowatt-hours of electricity each day.

Review Sheet #4
question #7

(7) $D(r) = 10{,}000(2-r)$

(a) $\lim_{r \to ?} 10{,}000(2-r) = 0 \ \therefore \ r = 2$ miles

(b) 



Area $= 2\pi r \Delta r$

(c) Population = Area x Density

$$= 2\pi r \Delta r \cdot 10{,}000(2-r)$$

(d) $$\int_0^2 10{,}000(2-r)(2\pi r)\,dr =$$

$$20{,}000\pi \int_0^2 (2-r)r\,dr = 20{,}000\pi \int_0^2 (2r-r^2)\,dr$$

$$20{,}000\pi \left(r^2 - \frac{r^3}{3} \right) \Big|_0^2$$

$$20{,}000\pi \left(4 - \frac{8}{3} \right) = \frac{80{,}000\pi}{3} \text{ people}$$

The population in the city is approximately 83776 people.